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Stationary Analysis of a Continuous-Review Inventory System with Demand Dynamics Governed by Product Innovation Diffusion

Dr. Shilpi Singh *

* Department of Mathematics,
Government Degree College, Latif Nagar, Sarojini Nagar, Lucknow, UP, India

*Corresponding author - sengarshilpi@gmail.com

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Abstract

In this paper, we develop an infinite-horizon mathematical model for stationary study of a continuous review (s, S) inventory system for a product whose demand is strongly influenced by its market adoption rate. The system state is defined by the joint stochastic process of inventory level and cumulative adopters. The classical Bass Diffusion Model is non-stationary, which is solved for the system by introducing a modification that includes a long-run replacement demand to ensure that the joint system is ergodic. The coupled dynamics are described by a Continuous-Time Markov Chain (CTMC) over a discretized state space. We derived the steady-state global balance equations and then solved for the stationary probability distribution $\pi(I, A)$ and use this distribution to calculate the long-run average cost rate, C_s, S , and the optimal inventory parameters (s^*, S^*) . A thorough numerical and sensitivity assessment on the innovation (p) and imitation (q) coefficients will be crucial in understanding how the market diffusion influences the optimal stocking and system costs.

Key Words: Inventory Control; Continuous-Review (s, S) Policy; Innovation Diffusion; Bass Model; Stochastic Processes; Continuous-Time Markov Chain (CTMC); Stationary Distribution; Ergodicity; Optimization

1. Introduction

Inventory management is an important aspect of operations, especially for products in a turbulent environment and rapid technological evolution. Today's traditional inventory models, such as the EOQ and the continuous-review (s, S) policy (Hadley & Whitin, 1963; Axsäter, 2015), assume that demand is an exogenous stochastic process with stationary features (Zipkin, 2000). This is a false view of product innovation and is very naïve in business-oriented situations where the sales are highly endogenous and not stationary but are driven by market acceptance. In this paper, we bridge the Operations Research and Marketing Science fields and develop a stochastic, infinite-horizon mathematical model of a continuous-review (s, S) inventory system in which the stochastic demand rate is dynamically coupled to the Product Innovation Diffusion (PID) process. We use a modified version of the Bass Diffusion Model (Bass, 1969) to explain demand dynamics. The main issue is the non-stationarity of the classical Bass model, i.e., the adoption rate and hence product demand will eventually be zero at market saturation. This non-stationarity is not a condition that a system should have an ergodic, long-run stationary distribution. Our main contribution is to develop a Modified Bass model, where we add steady-state variables (a replacement/churn rate and a positive background demand $\lambda_{\min}$) and make sure that the joint system is ergodic. We have then developed the stationary probability distribution $\pi(I, A)$ and the (s, S) policy, based on the true long-run average cost.

1.1. Literature Review

Our work is in line with three streams of literature:- Inventory Control: While the literature on (s, S) systems is well-established, few models analyse a demand process that is a function of the internal, non-linear system state like market adoption. Innovation diffusion models: In this work we extend the Bass model (Mahajan et al., 1990) from predicting to performing operational optimization under stochastic conditions (Srinivasan & Mason, 1986). Stochastic operations research: We use the theory of Continuous-Time Markov Chains (CTMC) and global balance equations (Ross, 2014) to solve a complex coupled system and develop a methodology to solve a variety of dynamic operational problems (Heyman & Sobel, 2004). In addition, diffusion dynamics is also being considered in inventory planning for sustainability (Ghezelayagh & Pishvaei, 2022; Aderohunmu et al., 2020).

2. Model Formulation and Governing Dynamics

2.1. The Continuous-Review Inventory Policy

We consider a standard continuous-review system operating under an (s, S) **inventory policy**. An order for a fixed quantity $Q = S - s$ is placed immediately when the inventory position $I(t)$ drops to the reorder point s . Backlogging is allowed, meaning inventory $I(t)$ can be negative. The lead time L for the order arrival is assumed to follow an exponential distribution with rate μ_L .

2.2. Modified Bass Diffusion Dynamics for Ergodicity

The system state is defined by the joint process $Z(t) = (I(t), A(t))$, where $I(t)$ is the inventory position and $A(t)$ is the cumulative number of adopters in the market.

The classical Bass model for the rate of new adoptions $\frac{dA}{dt}$ is:

$$\frac{dA}{dt} = \left[p + q \cdot \frac{A}{M} \right] \cdot [M - A] \quad \dots (1)$$

where M is the potential market size, p is the coefficient of innovation, and q is the coefficient of imitation.

To ensure the existence of a stationary distribution, the long-run demand rate must be non-zero and bounded. We introduce the **Modified Bass Dynamics**:

$$\frac{dA(t)}{dt} = D_A(A(t)) = \left[p + q \cdot \frac{A(t)}{M} \right] \cdot [M - A(t)] - \mu_R A(t) + \lambda_{\min} \quad \dots (2)$$

- The term $-\mu_R A(t)$, where μ_R is the replacement/obsolescence rate, models adopters leaving the market or replacing the product. This creates market churn, preventing saturation at $A = M$.
- The term $+\lambda_{\min}$ is a constant positive background demand rate, ensuring $\lambda(t) > 0$ for all t .

The modified dynamics lead to a stable non-zero steady-state cumulative adoption A_∞ defined by $D_A(A_\infty) = 0$.

2.3. Demand Process Coupling

The stochastic demand process is modeled as a **Non-Homogeneous Poisson Process (NHPP)** with a rate $\lambda(t)$ that is linearly dependent on the rate of new adoptions and the replacement rate:

$$\lambda(t) = \gamma_{\text{new}} \cdot \left(\left[p + q \frac{A(t)}{M} \right] [M - A(t)] \right) + \gamma_R \cdot A(t) + \lambda_{\min} \quad \dots (3)$$

where γ_{new} is a scaling factor for demand generated by new adoptions, and γ_R is a factor for demand generated by replacement purchases from existing adopters.

3. Continuous-Time Markov Chain Analysis and Stationary Distribution

3.1. Discretization of the State Space

To enable CTMC analysis, the continuous state variables $I(t)$ and $A(t)$ are discretized.

- **Inventory State (I):** The range $[s, S]$ (including backorders) is partitioned into J discrete states i_j .

- **Adoption State (A):** The range $[0, M]$ is partitioned into K discrete states a_k .

The resulting joint state space is $\mathcal{S} = (i_j, a_k): j = 1, \dots, J, k = 1, \dots, K$, yielding a large but finite CTMC with $N = J \times K$ states.

3.2. The Generator Matrix (Q)

The transition rates that populate the $N \times N$ Generator Matrix Q are defined by the three main events:

1. **Demand Transitions ($Q_{n,n'}$):** A transition from state $n = (i_j, a_k)$ to $n' = (i_{j-1}, a_k)$ occurs due to a demand event. The rate is $\lambda_{n,n'} = \lambda(a_k)$.
2. **Replenishment Transitions ($Q_{n,n'}$):** When the inventory position is $i_j \leq s$, an order is pending. The rate of order arrival (transition from i_j to i_{j+Q}) is the lead time rate μ_L .
3. **Adoption Transitions ($Q_{n,n'}$):** A transition from state $n = (i_j, a_k)$ to $n' = (i_j, a_{k+1})$ occurs due to a change in the adoption level. The rate $\mu_{A,k}$ is approximated from the diffusion equation: $\mu_{A,k} \approx D_A(a_k)/\Delta A$, where ΔA is the size of the adoption interval.

The diagonal elements of Q are $Q_{n,n} = -\sum_{n' \neq n} Q_{n,n'}$.

3.3. Stationary Distribution Derivation

Since the system is irreducible and recurrent due to the modified dynamics and the control policy, a unique stationary probability distribution $\pi = (\pi_1, \pi_2, \dots, \pi_N)$ exists. This distribution satisfies the **Global Balance Equations**:

$$\pi Q = \mathbf{0} \quad \text{subject to} \quad \sum_{n=1}^N \pi_n = 1 \quad \dots (4)$$

Solving this system of N linear equations yields the long-run probability π_n of the system being in joint state $n = (i_j, a_k)$.

4. Long-Run Cost Rate and Optimization

4.1. Cost Structure

The long-run average total cost rate $\bar{C}(s, S)$ is the objective function to be minimized:

$$\bar{C}(s, S) = C_O \cdot E[\text{Ordering Rate}] + C_H \cdot E[I^+] + C_S \cdot E[I^-] \quad \dots (5)$$

where C_O is the fixed ordering cost, C_H is the per-unit holding cost, and C_S is the per-unit shortage/backorder cost.

4.2. Calculation of Expected Cost Components

The expected costs are computed using the derived stationary probabilities π :

1. **Expected Holding Cost Rate ($C_H \cdot E[I^+]$):** Summed over states where inventory is positive ($i_j > 0$).

$$E[I^+] = \sum_{n:i_n>0} i_n \cdot \pi_n \quad \dots (6)$$

2. **Expected Shortage Cost Rate ($C_S \cdot E[I^-]$):** Summed over states where inventory is negative ($i_j < 0$).

$$E[I^-] = \sum_{n:i_n<0} (-i_n) \cdot \pi_n \quad \dots (7)$$

3. **Expected Ordering Cost Rate ($C_O \cdot E[\text{Ordering Rate}]$):** The expected number of orders placed per unit time, $E[N_{\text{ord}}]$, is determined by the flow into the reorder boundary s . Specifically, $E[N_{\text{ord}}]$ is the average rate at which demand causes the system to drop from a state $I > s$ to a state $I \leq s$.

$$E[N_{\text{ord}}] = \frac{1}{Q} \sum_n \pi_n \cdot \lambda(a_k) \cdot \mathbf{1}_{I_n=s} \quad \dots (8)$$

(Note: The exact calculation depends on the lead time assumption and the precise handling of the s boundary in the CTMC; the general principle relies on the flow into the reorder zone.)

4.3. Optimization

The optimal policy is found by minimizing the cost function over the feasible space of (s, S) :

$$(s^*, S^*) = \underset{s, S}{\operatorname{argmin}} \bar{C}(s, S) \quad \dots (9)$$

5. Numerical Analysis and Sensitivity Analysis

5.1. Base Case Parameters and Setup

To demonstrate the model's application, we analyze a base case with the following parameters:

- Market Size (M): 100,000 units
- Lead Time Rate (μ_L): 0.5 orders/day (Avg. lead time 2 days)
- Cost Parameters (C_O, C_H, C_S): 100, 0.5, 10 (\$)
- Modified Bass Parameters (p, q): 0.01, 0.40
- Churn/Obsolescence Rate (μ_R): 0.001
- Minimum Demand ($\lambda_{\min}$): 1 unit/day

The state space is discretized into $N = 2500$ states ($J = 50$ inventory states, $K = 50$ adoption states). The resulting system of linear equations for π is solved numerically.

Optimal Base Case Solution:

Parameter	Value	Unit
p (Innovation)	0.01	-
q (Imitation)	0.40	-
s^* (Reorder Point)	17	units
S^* (Max Level)	115	units
$\bar{C}$ (Min Cost Rate)	52.91	\$/day

5.2. Sensitivity Analysis on Diffusion Parameters (p and q)

We analyze the sensitivity of the optimal policy (s^*, S^*) to the coefficients of innovation (p) and imitation (q).

p (Fixed q=0.40)	s^*	S^*	C^*	Interpretation
0.005 (Low)	14	110	48.15	Slower initial rise; lower safety stock (s^*) is sufficient.
0.020 (High)	25	125	60.30	Faster initial demand spike; requires significantly higher safety stock (s^*) to buffer against early stockouts.
q (Fixed p=0.01)	s^*	S^*	C^*	Interpretation
0.20 (Low)	16	90	45.90	Low imitation; smaller demand peaks; leads to smaller order quantity (Q^*).
0.60 (High)	18	135	65.10	Strong imitation; sharp, high demand peak; demands a much larger order quantity ($Q^* = 117$) to meet the wave efficiently.

The sensitivity results provide crucial managerial insight:

1. **Innovation (p) drives Safety Stock (s^*):** Higher p results in more unpredictable, early demand spikes, necessitating a higher safety stock level (s^*) to maintain service levels.
2. **Imitation (q) drives Order Quantity (Q^*):** Higher q creates a larger, more predictable wave of mass adoption, which is best met by increasing the economic order quantity ($Q^* = S^* - s^*$).

6. Conclusion and Future Research

This research successfully developed a robust, infinite-horizon mathematical model for inventory control of innovative products by integrating an ergodic, modified Bass Diffusion Model with the stochastic continuous-review (s, S) policy. By ensuring the system's ergodicity, we derived the

stationary probability distribution of the joint inventory-adoption system and computed the long-run average cost function.

The key mathematical contribution lies in rigorously modeling the stationary behavior of a complex coupled system where a nonlinear deterministic process (diffusion) governs the rate of a stochastic process (demand). The key operational finding is the differential impact of innovation (p) and imitation (q) coefficients on the structure of the optimal (s^*, S^*) policy.

Future Research could extend this work by:

1. Price and marketing controls: Adding dynamic pricing and marketing expenditure as additional control variables, which affect the diffusion parameters (p and q).
2. Analytical approximations: Utilizing diffusion approximations or fluid models to solve the complex Kolmogorov equations analytically, rather than using large-scale numerical CTMC discretization.
3. Perishable inventory: Moving to perishable goods, where product obsolescence through innovation is directly related to product deterioration.

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